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On Lagrangian drift in shallow-water waves on moderate shear

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The Lagrangian drift in an $O(\epsilon)$ monochromatic wave field on a shear flow, whose characteristic velocity is $O(\epsilon)$ smaller than the phase velocity of the waves, is considered. It is found that although shear has only a minor influence on drift in deep-water waves, its influence becomes increasingly important as the depth decreases, to the point that it plays a significant role in shallow-water waves. Details of the shear flow likewise affect the drift. Because of this, two temporal cases common in coastal waters are studied, viz. stress-induced shear, as would arise were the boundary layer wind-driven, and a current-driven shear, as would arise from coastal currents. In the former, the magnitude of the drift (maximum minus minimum) in shallow-water waves is increased significantly above its counterpart, viz. the Stokes drift, in like waves in otherwise quiescent surroundings. In the latter, on the other hand, the magnitude decreases. However, while the drift at the free surface is always oriented in the direction of wave propagation in stress-driven shear, this is not always the case in current-driven shear, especially in long waves as the boundary layer grows to fill the layer. This latter finding is of particular interest vis-à-vis Langmuir circulations, which arise through an instability that requires differential drift and shear of the same sign. This means that while Langmuir circulations form near the surface and grow downwards (top down), perhaps to fill the layer, in stress-driven shear, their counterparts in current-driven flows grow from the sea floor upwards (bottom up) but can never fill the layer.

Key words: ocean processes, shallow-water flows, waves/free-surface flows

1. Introduction

Surface gravity waves interact with themselves to realize a mean drift, usually in the direction of wave propagation. Known as the Stokes drift (Stokes 1847) in irrotational waves travelling through otherwise quiescent fluids, and sometimes referred to as the generalized Stokes drift when the waves further interact with a supporting shear flow (Andrews & McIntyre 1978; Phillips 2001*a*), the drift is a Lagrangian motion which provides a measure of the mean particle velocity (relative to any mean supporting flow). Because of this, it is further known as the Lagrangian drift and also wave drift. Being an averaged quadratic (wave–wave) property, waves of slope ϵ necessarily realize a drift of $O(\epsilon^2)$, which is typically small. Nevertheless, the drift

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plays key roles in a variety of physical phenomena, which include destabilizing surface waves (Caponi *et al.* 1991), deforming the leading edge of oil slicks (Christensen & Terrile 2009) and affecting the appropriate alignment of vorticity to form Langmuir circulations (Craik & Leibovich 1976). Appropriately, therefore, drift has been the topic of numerous studies, albeit predominantly in irrotational waves in near quiescent surroundings (e.g. Stokes 1847; Longuet-Higgins 1953; Craik 1982*a*). Algorithms to extract drift from ocean data have also been derived (Smith 1992; Phillips 2001*b*; Smith 2006). Of course, drift is not confined to incompressible free-surface waves, and can likewise arise in wave fields induced by a flow over rigid wavy walls, be the flow unidirectional (Craik 1982*c*; Phillips & Wu 1994) or oscillatory (Larrieu, Hinch & Charru 2009). Drift can also arise in acoustic waves in a compressible flow, where it is denoted by acoustic or steady streaming (Lighthill 1978; Rayleigh 1883).

Ubiquitous to the above references is that drift is a wave property, in the sense that it would not occur in the absence of waves. But a lack of consistency in defining what drift is beyond this continues to be the source of much confusion. Here we define 'drift' as the difference between two well-defined mean velocities, the Lagrangian mean and the Eulerian mean, a definition that arises naturally in the generalized Lagrangian-mean (GLM) formulation of Andrews & McIntyre (1978).

This definition is at once commensurate with the original work by Stokes (1847) for irrotational waves in quiescent surroundings but does not always concur with the usage by others when the surroundings are other than quiescent. For example, waves commonly occur in combination with wave-induced or pre-existing shear flows. Longuet-Higgins (1953) exposed the former by showing that mass transport in the interior is composed of the Stokes drift (as it arises in an inviscid situation) plus a wave-induced transient correction of the same order that is a consequence of viscous boundary effects. Under our definition, the drift is solely the Stokes drift and the transient correction is an induced Eulerian-mean velocity, while others interpret the whole wave-induced flow as 'drift'. Rotational effects likewise affect the mass transport (Ursell 1950; Hasselmann 1970, 1971) and give rise to a further, cross-wave component of flow, also of the order of the Stokes drift (Xu & Bowen 1994). Again, by our definition, the drift is solely the Stokes drift, whereas the authors of the study view the whole induced flow field as 'drift'. Of course, the Lagrangian-mean flow is invariant whichever definition is used; all that is different is the terminology of components which comprise this Lagrangian mean. From here on, we use the term 'drift' only in the context of our definition.

To highlight the role of pre-existing shear, Craik (1982*c*) studied drift in monochromatic plane waves induced by (strong: see below) uniform shear over rigid wavy walls. Here the waves are irrotational. But should the shear be non-uniform (and strong), the waves are rotational. Drift in this case was first investigated by Phillips & Wu (1994) by also employing monochromatic plane waves induced by rigid wavy walls (see also Phillips & Shen 1996; Phillips, Wu & Lumley 1996). Phillips (2001*a*) went on to consider drift in discrete and continuous spectra of rotational waves and (Phillips 2005) in free-surface waves, all in strong shear.

To place the role of shear in perspective, consider two-dimensional monochromatic straight-crested waves of phase velocity $\mathscr{C}$ on a unidirectional mean shear flow of characteristic velocity $\mathscr{V}$. We align x in the direction of wave propagation, which is also the direction of the mean flow, and set z vertical (positive upwards) with y in the cross-stream direction. Further, we denote the wavenumber of the waves by $\mathfrak{K}$ and their wave amplitude by $\mathfrak{a}$ such that their wave slope $\epsilon = \mathfrak{K}\mathfrak{a}$; accordingly, since the role of the layer depth $\mathfrak{h}$ is of interest, we let $\alpha = \mathfrak{K}\mathfrak{h}$. Then, since orbital velocities in the wave field are characterized by $\epsilon\mathscr{C}$, we find that the vertical gradient of mean velocity or shear scales as $\mathscr{V}/\mathscr{C} = O(\epsilon^s)$, where $s \in [0, 2]$ is the shear index (Phillips 1998, 2003). The indices $s = 0, 1, 2$ concur with usage by Craik (1982*c*) of 'strong', 'moderate' and 'weak' shear.

Thus, in the context of wind-driven surface waves, strong shear indicates that the ratio of the surface velocity to the wave phase velocity is $O(1)$, as is the case in wind-driven laboratory waves (Veron & Melville 2001). Oceanic waters, on the other hand, experience a variety of levels of shear (Melville, Shear & Veron 1998) and, because of this, we use the phase speed of the dominant waves to specify what is then a typical level of shear. We then find that the shear in the open ocean is $s = 2$ or weak (Craik & Leibovich 1976), while shear in coastal waters is $s = 1$ or moderate (Marmorino, Smith & Lindemann 2005).

The question, of course, is how the drift is influenced by s? To resolve this we restrict attention to small-amplitude waves in which, from GLM theory (see §2), the drift $\boldsymbol{d}$ takes the form (Andrews & McIntyre 1978)

$$d_i = \overline{\xi_j \breve{u}_{i,j}} + \tfrac{1}{2}\overline{\xi_j \xi_k}\,\overline{u}_{i,jk} + O(\epsilon^3). \tag{1.1}$$

(Here, indices $(1, 2, 3) \mapsto (x, y, z)$, repeated indices imply summation and commas denote partial differentiation.) The variable $\bar{\boldsymbol{u}}$ is the mean Eulerian velocity, $\breve{\boldsymbol{u}}$ is the Eulerian fluctuating velocity and $\boldsymbol{\xi}$ is the associated particle displacement field. Since the first term in (1.1) is not affected by shear, we identify it with the Stokes drift and, in fact, will use the term Stokes drift only in the context of irrotational waves in near quiescent surrounds. Moreover, in analysing the first term we would typically normalize in terms of the wave properties $\mathscr{C}$ and $\mathfrak{K}$, but to estimate the importance of the second term in (1.1) the mean velocity and z are more appropriately scaled in terms of $\mathscr{V}$ and $\mathfrak{h}$. Then, because (Phillips 1998)

$$\bar{\boldsymbol{u}} = \epsilon^s [U(z, t), 0, 0], \tag{1.2}$$

we see that the ratio of the second to the first terms is $O(\epsilon^s \alpha^{-2})$.

In other words, the second term is of the order of the first whenever $\epsilon^s \alpha^{-2} = O(1)$, which, in turn, necessitates that

$$\frac{\mathfrak{a}}{\mathfrak{h}} = O\left(\epsilon^{(2-s)/2}\right). \tag{1.3}$$

Of course, (1.3) is also subject to the physical requirement $\mathfrak{a}/\mathfrak{h} \ll 1$, which, in turn, necessitates that $s = 0$ or $s = 1$, not $s \geqslant 2$. In short, the second term in (1.1) must be retained when $s \in [0, 1]$.

Therefore, in the context of previous work, we see that while the aforementioned studies by Stokes (1847), Longuet-Higgins (1953) and others are in the $s = 2$ category, those by Craik, Phillips and their colleagues are for $s = 0$. Curiously, the $s = 1$ case when $\epsilon\alpha^{-2} = O(1)$ would appear not to have been attempted and this is our focus here.

In truth, our interest in drift in $s = 1$ shear really stems from an interest in Langmuir circulations in coastal waters, the connection being that Langmuir circulations arise through an instability that presumes not only shear and differential drift, but also that they be of the same sign (Craik & Leibovich 1976; Phillips 2003). Langmuir circulations are wind-aligned rolls near the ocean surface that can grow in cross-section to the size of sports stadiums and are crucial to the formation and maintenance of the upper ocean surface-mixed layer (Langmuir 1938; Craik & Leibovich 1976; Phillips 2002; Babanin, Ganopolski & Phillips 2009). In contrast, in coastal waters, they are known to penetrate to the ocean floor and, in doing so, to vastly enhance the level of sediment mixing (Gargett *et al.* 2004). Therefore, as a precursor to studying Langmuir circulations in shallow-water credibly, we require the drift in $s=1$ shear. We further note that the vertical structure of the flow in coastal waters is also of interest to the broader wave/current interaction community and that, to this point, almost no details are known on the effect of higher-order corrections to the lowest-order Stokes drift, and/or the effect of weakly rotational waves.

The physical setting we have in mind then is one which includes, at the deeper end, that observed by Gargett *et al.* (2004) (off the coast of New Jersey), in which the water is circa 15 m in depth and the wavelength of the dominant waves circa 100 m, to much shallower water, say 2 m in depth, as in the observations of Marmorino *et al.* (2005) (in the Egmont Channel at the mouth of Tampa Bay). In fact, water depths less than this are admissible provided near-shore processes are minimal and the long waves do not violently break, although 2 m is the rule-of-thumb depth for the surf zone in which violent wave breaking and other near-shore processes dominate. On assuming $\epsilon=0.1$ then, we readily find that the requirement $\epsilon^s\alpha^{-2}=O(1)$ is met for $\mathfrak{h}=O(5\,\mathrm{m})$, that is $1.5\,\mathrm{m} \lesssim \mathfrak{h} \lesssim 15\,\mathrm{m}$ when $s=1$ and for $\mathfrak{h}=O(9\,\mathrm{m})$ or, say $3\,\mathrm{m} \lesssim \mathfrak{h} \lesssim 30\,\mathrm{m}$, when $s=1/2$. This range of depths indicate that our analysis is valid at the shallower end of what physical oceanographers denote as the inner coastal region.

In this region, over what is typically a 10 km length scale (known as the baroclinic deformation radius $\mathscr{B}$, say), the ocean floor gradually falls away to circa 200 m (beneath the free surface). Accordingly, this slow change in topography introduces a commensurate horizontal coordinate dependence (in x, say) on primary surface waves that evolve on horizontal scales significantly smaller than $\mathscr{B}$ (Chu & Mei 1970). That notwithstanding, and because the wavelength of the dominant waves is of $O(\epsilon^2\mathscr{B})$, it turns out that the leading-order results coincide with the linear horizontal solution and that topographic variations, and effects consequent to them, play a role only at higher orders. Indeed, for consistency, in their careful asymptotic expansion of variables pertinent to wave–current interactions in the inner coastal region, McWilliams, Restrepo & Lane (2004) determined that long-wave and current velocities must be entirely horizontal up to $O(\epsilon^3)$. Thus, because the flow fields considered in the present work lie within this range, we exclude topographic effects.

The flow field $\boldsymbol{u}(x,y,z,t)$ is thus composed of a wave field $\breve{\boldsymbol{u}}(x,z,t)$ and a mean flow $\bar{\boldsymbol{u}}(z,t)$, which, in appropriate circumstances, interact to spawn spanwise-dependent structures such as Langmuir circulations $\tilde{\boldsymbol{u}}(y,z,t)$. Of course, each flow component has its own velocity and time scale and they must together satisfy the Navier–Stokes (NS) equations. Furthermore, because our mindset is that the time scale of the evolving structure well exceeds that of the wave field, it requires us, when seeking evolution equations describing a bifurcation to $\tilde{\boldsymbol{u}}$, to average over the wave period (or wavelength) and retain input from the wave field only through nonlinear contributions resulting from the wave–wave interaction. This premise, along with restrictions that the wave field is irrotational and the base flow is weak ($s=2$), was employed by Craik & Leibovich (1976) in deriving their well-known Craik–Leibovich (CL) equations for $\tilde{\boldsymbol{u}}$.

When the shear is strong ($s=0$), on the other hand, the evolving structure $\tilde{\boldsymbol{u}}$ acts to distort the wave field and, since mean-field equations cannot capture this effect, they must be solved in tandem with a set that can capture it, viz. the Euler or the NS equations (Craik 1982*c*; Phillips 1998). But to proceed in tandem, the mean-field equations must depict the same conservative properties as their non-averaged counterparts and mean-field theories constructed from simple averaging (viz. temporal or spatial) satisfy this requirement only when the wave field is irrotational (Phillips 2001*a*). Indeed, to satisfy this requirement when the wave field is rotational, an appropriate diffeomorphic mapping (of the instantaneous equations) must be employed, as Andrews & McIntyre (1978) do in their GLM formulation. It is for this reason that Phillips (1998) employed GLM in tandem with the NS equations to derive a general set of evolution equations for $\tilde{\boldsymbol{u}}$ that are valid in rotational waves for all levels of shear $s \in [0, 2]$. This set, denoted by the CL-generalized (CLg) equations, is outlined in §2 and recovers the CL equations when $s = 2$ and their inviscid counterparts by Craik (1982*c*) when $s = 0$.

In fact, the CLg equations represent an asymptotic theory for the interaction of wavy shear flows in generic circumstances and were used by Phillips (2005) to expose the instability mechanism responsible for the formation of Langmuir circulations in the ($s = 0$) laboratory experiments of Melville *et al.* (1998) and Veron & Melville (2001). McWilliams *et al.* (2004), on the other hand, provide a rational asymptotic theory specifically tailored to the interaction of approximately irrotational waves and currents in coastal waters, with slow dependencies on topography, long waves and currents. They too average the primitive equations over the primary-wave oscillations and, as in the GLM formulation, CLg and CL equations, they also highlight the importance of the term resulting from averaged wave–wave nonlinearities. Two equivalent representations of this term are possible: a radiation stress (known in some communities as the Reynolds stress) and a Bernoulli head plus a force term; Lane, Restrepo & McWilliams (2007) discuss the benefits of each in detail. Of specific interest to us is the latter form, first because it includes the drift and second because it is central to describing an instability mechanism to Langmuir circulations.

To proceed, therefore, we employ GLM theory to derive an appropriate mean field theory, an outline of which is given in §2, in tandem with the NS equations, which we employ to provide details of the shallow-water wave field (in §3). We calculate the drift in §4. Since the drift further depends on details of the base flow, temporal profiles of such are derived in §5 for two typical cases, viz., wind- or surface-stress-induced shear and current-induced shear. Our results are discussed in §6.

2. Generalized Lagrangian mean theory

Deriving the drift in rotational waves that propagate on a mean shear flow is somewhat more complicated than deriving the Stokes drift and, because of this, it is best to proceed from within the consistent framework of Andrews and McIntyre's GLM theory (Andrews & McIntyre 1978).

GLM theory is an exact and very general Lagrangian-mean description of the back effect of oscillatory disturbances upon the mean state and is thus valid for waves of all amplitudes, although, for practical purposes, it has so far been restricted to waves of small amplitude, measured by the small parameter ϵ, so that any displacement $\boldsymbol{\xi}$ from the mean trajectory is $O(\epsilon)$ compared to the wavelength of the wave field.

GLM theory is compelling because it satisfies (in inviscid flow) a Kelvin-like theorem akin to that satisfied by the Euler equations. This means that GLM exhibits, in the mean, the same conservative properties as does Euler, be the waves rotational or irrotational, and thus, that the instantaneous and averaged equations can be solved simultaneously: the NS/Euler equations to determine admissible wave fields and any modulation to them caused by secondary motions, and GLM the effect of said wave field on the mean flow (see e.g. Craik 1982*c*; Phillips & Wu 1994; Phillips & Shen 1996; Phillips 1998, 2005).

In constructing GLM theory, we must first define an exact Lagrangian-mean operator $\overline{(\ )}^L$ corresponding to any given Eulerian-mean operator $\overline{(\ )}$. This necessitates defining with equal generality at location $\boldsymbol{x}$ and time t an exact, disturbance-associated particle-displacement field $\boldsymbol{\xi}$ which has zero mean when any average, be it temporal, spatial or ensemble, is applied. For any scalar or tensor field φ, it is then possible to write

$$\overline{\varphi(\boldsymbol{x},t)}^L = \overline{\varphi^\xi(\boldsymbol{x},t)}, \quad \text{where} \quad \varphi^\xi(\boldsymbol{x},t) = \varphi(\boldsymbol{x}+\boldsymbol{\xi},t).$$

Then, provided the mapping $\boldsymbol{x} \mapsto \boldsymbol{x}+\boldsymbol{\xi}$ is diffeomorphic and provided $\overline{\boldsymbol{\xi}(\boldsymbol{x},t)}=0$, there is a unique Lagrangian-mean velocity $\overline{\boldsymbol{u}}^L$ related to any given Eulerian-mean velocity $\overline{\boldsymbol{u}}(\boldsymbol{x},t)$ by the drift $\boldsymbol{d}$, as $\overline{\boldsymbol{u}}^L = \overline{\boldsymbol{u}}+\boldsymbol{d}$. Of course, the Lagrangian-mean velocity is synonymous with the mass-transport velocity and if we view the findings of Longuet-Higgins (1953) in light of this, we interpret his drift component as solely the inviscid Stokes drift and his transient flow as a wave-induced Eulerian-mean velocity; and thus in Xu & Bowen (1994) also, where their cross-wave component resulting from rotational effects is not drift but again an Eulerian-mean velocity. Thus, since rotational effects would appear to be largely confined to the Eulerian-mean velocity and not the drift, we will, at this stage, exclude them.

In fact, two mean quadratic measures of the nonlinear interaction of the fluctuations with themselves and supporting shear flow arise in GLM theory, the drift $\boldsymbol{d}$ and the pseudomomentum $\boldsymbol{p}$, the latter relaxing to the former when the waves are irrotational (Andrews & McIntyre 1978; Phillips 2001*a*). In order to isolate the drift, however, we must first introduce the Lagrangian-mean material derivative $\overline{D}^L = \partial/\partial t + \overline{\boldsymbol{u}}^L \bullet \nabla$, from which it follows that $\overline{D}^L \boldsymbol{\xi} = \boldsymbol{u}^l$, where the Lagrangian velocity perturbation $\boldsymbol{u}^l$ satisfies $\overline{\boldsymbol{u}}^l = 0$. It further follows that $\boldsymbol{u}^l$ may be written as $u^l_j = \breve{u}_j + \xi_k \overline{u}_{j,k} + O(\epsilon^2)$, from which we find that the pseudomomentum $p_i = -\overline{\xi_{j,i} u^l_j}$ and, for small-amplitude waves, that the drift is as given in (1.1).

2.1. *The generalized Lagrangian-mean equations*

For homentropic flows of constant density ρ in a non-rotating reference frame, the GLM momentum equations, when written in a form akin to the NS equations, become (Phillips 1998)

$$\bar{q}_{i,t} + \bar{q}_j \bar{q}_{i,j} - p_j(\bar{q}_{j,i} - \bar{q}_{i,j}) + \Pi_{,i} = \mathscr{X}_i, \tag{2.1}$$

where $\bar{q}_i = \bar{u}^L_i - p_i$,

$$\Pi = \frac{\wp}{\rho} + \bar{\Phi}^L_i - \frac{1}{2}\overline{u^\xi_j u^\xi_j} + \frac{1}{2}\bar{q}_j\bar{q}_j + \bar{p}_j\bar{q}_j,$$

and $\mathscr{X}$ allows for dissipative forces as

$$\mathscr{X}_i = \nu\left[\bar{u}^L_{i,kk} + \overline{\xi_{j,i} u^\ell_{j,kk}}\right],$$

where ν is the kinematic viscosity. Observe that the force term in (2.1) is expressed as $\boldsymbol{p} \times \nabla \times \bar{\boldsymbol{q}}$, that is the cross product of the pseudomomentum and the vorticity-associated vector field $\mho = \nabla \times \bar{\boldsymbol{q}}$, and reduces to the familiar $\boldsymbol{d} \times \nabla \times \bar{\boldsymbol{u}}$ (sometimes denoted as the vortex force) when the waves are irrotational. Further, the pressure $\wp$ and force potential Φ (which is zero here), along with the Bernoulli head, are incorporated into Π and thus vanish when we take the curl of (2.1) to realize $\mho$, viz. (Phillips 1998)

$$\mho_{i,t} + (\bar{q}_j + \bar{p}_j)\mho_{i,j} = \mho_j(\bar{q}_i + \bar{p}_i)_{,j} - \mho_i(\bar{q}_j + \bar{p}_j)_{,j} + \varepsilon_{ijk}\mathscr{X}_{k,j}, \tag{2.2}$$

where ε_{ijk} is the alternating tensor.

2.2. *Imposed shear of specified strength and $O(\epsilon)$ waves*

Phillips (1998) employed (2.1) and (2.2) to investigate a class of unidirectional shear flows that had imposed on them, or were unstable to, small-amplitude waves of slope $O(\epsilon)$ whose drift/pseudomomentum signatures are independent of the spanwise direction. Herein $\mathfrak{h}$ is the characteristic thickness of the shear layer and, in accord with our discussion in § 1, we make variables dimensionless with respect to $\mathfrak{h}$ and $\mathscr{C}$. Then $\mathscr{V}/\mathscr{C} = O(\epsilon^s)$ and in the event viscosity plays a role, the Reynolds number $\mathscr{R} \equiv \mathfrak{h}\mathscr{C}/\nu$.

For clarity, we use upper-case letters to denote quantities pertaining to the primary flow, which, by design, is devoid of spanwise (y) dependence, and lower-case letters otherwise, while an overbar on the unscaled dimensionless variable denotes a streamwise average, typically over one wavelength. Our construct places no restrictions on the relative time scales of the evolving mean flow and wave field, but for definiteness, we adopt the mindset that they are disparate with the mean flow having the longer time scale.

Envisage now an $O(\epsilon)$ wave field $\breve{\boldsymbol{U}}$ that interacts with the primary shear flow to excite streamwise-averaged spanwise-varying Eulerian velocity perturbations $\tilde{\boldsymbol{u}}$, whose strength relative to the primary shear flow is measured by the parameter $\varDelta$, and express the resulting flow field in GLM variables. The outcome is the velocity-associated vector field $\bar{\boldsymbol{q}} = \bar{\boldsymbol{Q}} + \tilde{\boldsymbol{q}}$, which we expand as

$$\bar{\boldsymbol{q}}(y, z, t) = \epsilon^s\{[Q_1, 0, \epsilon^{2-s}Q_3] + \varDelta[q_1, \epsilon^n q_2, \epsilon^n q_3] + \cdots\} \quad (n \geqslant 0), \tag{2.3}$$

and an affiliated scalar field $\varPi = \epsilon^s[\mathscr{G}(x, z, t) + \varDelta g(x, y, z, t) + \cdots]$, which includes the pressure. Note that the power n can have values other than zero and that n is related to s, as we shall see shortly.

In the first instance, the waves produce $O(\epsilon^2)$ primary fields of pseudomomentum and drift. But because the $O(\epsilon^s\varDelta)$ axial-velocity perturbation (due to the interaction between the waves and mean flow) may, in turn, act to distort the wave field and produce an $O(\epsilon^{s+2}\varDelta)$ spanwise varying component of pseudomomentum (Craik 1982*c*) we expand $\boldsymbol{p}$ and $\boldsymbol{d}$ as

$$\bar{\boldsymbol{p}}(y, z, t) = \epsilon^2\{[P_1, 0, P_3] + \epsilon^s\varDelta[p_1, \epsilon^n p_2, \epsilon^n p_3 + \cdots]\}. \tag{2.4}$$

Finally, because the Eulerian- and Lagrangian-mean velocity fields are related through $\bar{\boldsymbol{q}} = \bar{\boldsymbol{u}} + \boldsymbol{d} - \boldsymbol{p}$, we see that $Q_3 = D_3 - P_3$, which explains the extra primary mean field component in (2.3) (in contrast to the primary Eulerian flow, which, by design, has only one component).

2.3. *Conservation of mass*

GLM flows are typically not divergence-free. Indeed, mass conservation requires that

$$\bar{D}^L J + J(q_i + p_i)_{,i} = 0, \tag{2.5}$$

where J is the Jacobian of the mapping $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{\xi}$. Nevertheless, because $\bar{u}_{i,i} = (\bar{q}_i + p_i - d_i)_{,i} = 0$, we are at liberty to introduce the perturbation Stream function ψ as

$$q_2 = \frac{\partial \psi}{\partial z} + \epsilon^2(d_2 - p_2) \quad \text{and} \quad q_3 = -\frac{\partial \psi}{\partial y} + \epsilon^2(d_3 - p_3). \tag{2.6}$$

2.4. *The primary-flow field*

We determine the initial primary-flow field by substituting (2.3) and (2.4) into (2.1). Then, because (2.3) must identically satisfy (2.1) with $q_1, q_2, q_3 = 0$ and because $\varPi$ here reduces to the mean streamwise body force G, the x-momentum equation takes the form (Phillips 1998)

$$\frac{\partial Q_1}{\partial \tau} + \epsilon^2 \mathscr{R} D_3 \frac{\partial Q_1}{\partial z} = -G + \frac{\partial^2}{\partial z^2}(Q_1 - \epsilon^{2-s}(D_1 - P_1)), \tag{2.7}$$

where $\tau = \mathscr{R} t$ and $G = \mathscr{R} \partial \mathscr{G} / \partial x$. We must also comment on the second-order term: Phillips (1998) analyses $\mathscr{X}_i$ in detail and determines that of its multiple parts, the component $\bar{u}_{i,kk}$ dominates in irrotational waves for all admissible s and in rotational waves for $s \leqslant 1$. These findings are reflected in (2.7).

We further note that because $Q_1 = U + \epsilon^{2-s}(D_1 - P_1)$, the primary flow is dominated by its Eulerian counterpart U for: (i) all admissible $s \leqslant 1$; (ii) neutral waves, because then $D_3 = 0$ and D_1, P_1 are time-independent; and (iii) irrotational waves because then $D_1 - P_1 = O(\epsilon^4)$ (Andrews & McIntyre 1978; Phillips 2001*b*). This is not to say that the wave field is impotent; on the contrary, in the presence of viscous boundary conditions the wave field induces an associated temporal Eulerian flow of the same order as the rectified wave field, viz. $s = 2$, as Longuet-Higgins (1953) determined long ago. External forces such as those imposed in § 5 do not negate this wave-induced Eulerian velocity, but because it manifests as a higher-order correction to U for $s \leqslant 1$, it plays no significant role in our calculation of drift.

2.5. *The secondary-flow field and the CLg equations*

To determine the secondary flow we again substitute (2.3) and (2.4) into (2.1), but this time subtract (3.7), which leads to the $O(\epsilon^s \Delta)$ streamwise evolution equation for q_1, while the same expansions and (2.2) yield the $O(\epsilon^{s+n} \Delta)$ streamwise component of the vorticity-associated vector field. For this general form we refer the reader to (Phillips 1998, § 4). Rather here, since our interest is with structure arising through an instability requiring $\partial P_1 / \partial z \neq 0$, we restrict attention to what Phillips (2003) refers to as the CLg equations, in which $n = (2-s)/2$ and time is rescaled as $\mathrm{t} = \epsilon^{(s+2)/2} t$, to yield

$$\frac{\partial q_1}{\partial \mathrm{t}} + \Delta\left(q_2 \frac{\partial q_1}{\partial y} + q_3 \frac{\partial q_1}{\partial z}\right) + \epsilon^{(2-s)/2} D_3 \frac{\partial q_1}{\partial z} + q_3 \frac{\partial Q_1}{\partial z} = \epsilon^{-(s+2)/2} \mathscr{R}^{-1} \nabla^2 q_1 + O\left(\epsilon^{(2-s)/2} \mathscr{R}^{-1}\right), \tag{2.8}$$

and

$$\frac{\partial \mho_1}{\partial \mathrm{t}} + \Delta\left(\frac{\partial \mho_1 q_2}{\partial y} + \frac{\partial \mho_1 q_3}{\partial z}\right) + \epsilon^{\frac{2-s}{2}} \frac{\partial}{\partial z}(\mho_1 D_3) + \frac{\partial q_1}{\partial y} \frac{\partial P_1}{\partial z} - \epsilon^s \frac{\partial Q_1}{\partial z} \frac{\partial p_1}{\partial y}$$
$$+ \epsilon^s \Delta\left\{\frac{\partial q_1}{\partial y} \frac{\partial p_1}{\partial z} - \frac{\partial q_1}{\partial z} \frac{\partial p_1}{\partial y}\right\} = \epsilon^{-(s+2)/2} \mathscr{R}^{-1} \nabla^2 \mho_1 + O\left(\epsilon^{(2-s)/2} \mathscr{R}^{-1}\right). \tag{2.9}$$

Note that because n varies with s, it is evident from (2.3) that transverse- and axial-velocity perturbations may differ in order, a point first made by Craik (1982*c*). Also, because wave distortion is reflected in p_1 and the term containing it in (2.9) is premultiplied by ϵ^s, it does not play a role in our case of interest, viz. $s = 1$. It does, however, play a significant role when $s = 0$, in which case a further equation must enter to complete the set (2.8) and (2.9); details are given in Phillips (1998) and examples in Phillips & Wu (1994), Phillips *et al.* (1996) and Phillips (2005). That notwithstanding, wave distortion in the y and z directions is $O(\epsilon^{3+(s/2)} \Delta)$ and may be neglected for all $s \in [0, 2]$, allowing us to rewrite (2.6) as

$$\frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z} = 0 \qquad \text{and thus} \qquad \mho_1 = -\nabla^2 \psi. \tag{2.10}$$

Of course, the CLg equations reduce to CL equations of Craik & Leibovich (1976) in the appropriate limit.

3. The wave field

To proceed we require the wave field given the mean shear. In fact, both are no more than admissible complimentary primary and secondary solutions to the equations of motion (i.e. the NS equations) subject to appropriate boundary conditions under the premise that the time scale of the mean shear well exceeds that of the waves. Of course, the two effectively decouple when $s=2$, as here the shear has negligible effect on the fluctuating part of velocity field, rendering the waves irrotational. When $s=0$, on the other hand, and the shear non-uniform, the wave field is rotational (Phillips & Wu 1994; Phillips & Shen 1996). Of interest in the present study are weakly rotational waves in moderate ($s=1$) shear, but because much of the following analysis carries over to values of s other than unity, we shall leave s as s.

Consider then a layer of liquid in which we identify $z=0$ with the mean free surface and $z=-1$ with the rigid bottom. Of interest is the y-independent wave field $\breve{\boldsymbol{u}}$, which is admissible as a secondary flow in the presence of a mean primary flow $\bar{\boldsymbol{u}}$, which together total the Eulerian velocity $\boldsymbol{u}$ as

$$\boldsymbol{u}(x,z,t)=\bar{\boldsymbol{u}}(z,t)+\breve{\boldsymbol{u}}(x,z,t). \tag{3.1}$$

3.1. *Primary flow*

Looking first to the mean flow and allowing it to evolve with time τ on a long scale with respect to the wave period, we find from (2.7) that

$$\frac{\partial U}{\partial \tau}=\frac{\partial^2 U}{\partial z^2}-G. \tag{3.2}$$

We shall derive relevant profiles for U in § 5, but of interest, right now, is the wave field.

3.2. *Wave equations*

The wave field is given by substituting (3.1) with (1.2) into the NS equations and then subtracting (3.2), although prior to making the substitution we decompose the wave field into $O(\epsilon)$ and $O(\epsilon^{s+1})$ components and, to fix ideas, restrict ourselves to monochromatic two-dimensional waves that satisfy continuity as

$$\breve{\boldsymbol{u}}=\epsilon[\Phi',0,-\mathrm{i}\alpha\Phi]\mathrm{e}^{\mathrm{i}\beta}+\epsilon^{s+1}[\phi',0,-\mathrm{i}\alpha\phi]\mathrm{e}^{\mathrm{i}\beta}. \tag{3.3}$$

Here, $\beta=\alpha x-\omega_0 t$ and, for generality, $\omega_0=\omega+\epsilon^s\omega_2$, where $\sigma=\mathscr{C}\omega_0/\mathfrak{h}$ is the wave frequency. Then, on cross-eliminating pressure and collecting the fundamental mode $\mathrm{e}^{\mathrm{i}\beta}$, we find, at successive orders, variants of the Orr–Sommerfeld equation. However, because our flow field is devoid of critical layers, we do not expect viscous eigensolutions to play a significant role, and thus take the inviscid limit, to find

$$\epsilon[\omega\Phi''-\omega\alpha^2\Phi]+\epsilon^{s+1}[\omega\phi''-\omega\alpha^2\phi+\alpha U''\Phi]=0. \tag{3.4}$$

From (3.4) then, the governing equations at successive orders are thus a shortened Rayleigh equation depicting $O(\epsilon)$ irrotational waves

$$\Phi''-\alpha^2\Phi=0, \tag{3.5}$$

and a further Rayleigh equation at $O(\epsilon^{s+1})$ depicting rotational waves

$$\phi'' - \alpha^2\phi = -\tfrac{\alpha}{\omega}U''\Phi. \tag{3.6}$$

3.3. *Free-surface and other boundary conditions*

In order to solve (3.5) and (3.6) we require boundary conditions at the free surface and a rigid base. Looking first to the free surface $z=\eta(x,t)$, we impose kinematic and dynamic boundary conditions as

$$\frac{\mathrm{D}(z-\eta)}{\mathrm{D}t} = 0 \quad \text{and} \quad \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla\mathfrak{p} + \boldsymbol{k}\mathfrak{g} \quad \text{on} \quad z=\eta, \tag{3.7}$$

where $\rho\mathscr{C}^2\mathfrak{p}$ is the pressure, $\mathscr{C}^2\mathfrak{h}^{-1}\mathfrak{g}$ is the gravity and $\boldsymbol{k}$ is the unit normal in the z direction. Further, since $\nabla\mathfrak{p}$ has no component lying along the free surface, then

$$\nabla\mathfrak{p} = \frac{\partial\mathfrak{p}}{\partial z}\nabla(z-\eta) \quad \text{on} \quad z=\eta.$$

So, on setting

$$\eta = \frac{\epsilon \mathrm{e}^{\mathrm{i}\beta}}{\alpha} + O(\epsilon^2),$$

using a Taylor series to express $\nabla\mathfrak{p}$ on $z=\eta$ in terms of variables on $z=0$ (see Phillips 2005) and noting that pressure is specified by the hydrostatic law $\mathfrak{p}=-\mathfrak{g}z$, we find from (3.7) the free-surface boundary conditions at each order.

Specifically at $O(\epsilon)$ on $z=0$, we find

$$\alpha\Phi = \tfrac{\omega}{\alpha} \quad \text{and} \quad \omega\Phi' = \mathfrak{g}, \tag{3.8}$$

while at $O(\epsilon^2)$ on $z=0$, we have

$$\alpha\phi = \frac{\omega_2}{\alpha} - U(0), \tag{3.9a}$$

and

$$\omega\phi' = \alpha U(0)\Phi' - \omega_2\Phi' - \alpha U'(0)\Phi. \tag{3.9b}$$

Finally, in view of our rigid bottom, we have on $z=-1$ at all orders that

$$\Phi = \phi = 0. \tag{3.10}$$

3.4. *Leading-order solution: shallow-water waves*

The solution to the $O(\epsilon)$ problem, namely (3.5) subject to boundary conditions (3.8), is straightforward, which yields (Longuet-Higgins 1953)

$$\Phi = \frac{\omega}{\alpha^2}\frac{\sinh\alpha(z+1)}{\sinh\alpha}, \tag{3.11a}$$

with

$$\omega = \sqrt{\mathfrak{g}\alpha\tanh\alpha}. \tag{3.11b}$$

On the other hand, the higher-order problem (3.6) is somewhat more difficult and is treated, albeit for boundary conditions simpler than (3.9), by Phillips & Shen (1996) and for boundary conditions in accord with (3.9) by Phillips (2005). For our present purposes, however, we shall discover in §4 that only the $O(\epsilon)$ solution is necessary for the $s=1$ problem at hand.

4. The drift in sheared shallow-water waves

Our task now is to determine the drift, but prior to doing so, it is necessary to ascertain the displacement field $\boldsymbol{\xi}(\boldsymbol{x}, t)$ given the wave field. To do so we first note that $\overline{D}^L \xi_j = \mathrm{d}\xi_j/\mathrm{d}t$ and subsequently that (Phillips 1998)

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = \breve{u}_j + \xi_k \overline{u}_{j,k}, \tag{4.1}$$

along mean trajectories

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \overline{\boldsymbol{u}}^L(\boldsymbol{x}, t).$$

Further, we envisage the mean Eulerian base flow $\overline{\boldsymbol{u}}$ to be in the x-direction and to be a function of z only (at least on a long scale with respect to the wave period), and suppose that the displacement $\boldsymbol{\xi}$ is small compared with the radius of curvature of $\overline{\boldsymbol{u}}$. Then from (4.1)

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = \breve{u}_j + \xi_3 \overline{u}_{1,3} \delta_{j1},$$

where δ_{ij} is the Kronecker delta, so that

$$\xi_j(\boldsymbol{x}, t) = \int_{t_0}^{t} \breve{u}_j \, \mathrm{d}p + \delta_{j1} \overline{u}_{1,3} \int_{t_0}^{t} \xi_3 \, \mathrm{d}p, \tag{4.2}$$

with the requirement (Andrews & McIntyre 1978) that $\xi_j(\boldsymbol{x}, t) = 0$ at time $t = t_0$.

Now from (3.3), the component fluctuating velocities can be generalized as

$$\breve{u}_1 = \epsilon \Phi' A(t) \cos\beta + O(\epsilon^{s+1}), \tag{4.3a}$$

$$\breve{u}_3 = \epsilon \alpha \Phi A(t) \sin\beta + O(\epsilon^{s+1}), \tag{4.3b}$$

where $\mathfrak{a}A(t)$ is the wave amplitude. Bear in mind that although $A \sim 1$ for sufficiently large t, rendering the waves neutral, the waves grow to that amplitude from infinitesimal, namely $A(t_0) \to 0$ as $t_0 \to -\infty$ albeit over a time scale large compared with ω^{-1}. We shall see the importance of this shortly. From (4.2) the displacement fields then follow as

$$\xi_1 = \epsilon \Phi' H + \epsilon^{s+1} \alpha U' \Phi K + O(\epsilon^{2s+1}), \tag{4.4a}$$

$$\xi_3 = \epsilon \alpha \Phi I + O(\epsilon^{s+1}), \tag{4.4b}$$

with Lagrangian integrals

$$H(x, t) = \int_{t_0}^{t} A(p) \cos\beta \, \mathrm{d}p, \quad I(x, t) = \int_{t_0}^{t} A(p) \sin\beta \, \mathrm{d}p, \tag{4.5a}$$

and

$$K(x, t) = \int_{t_0}^{t} \int_{t_0}^{\zeta} A(p) \sin\beta \; \mathrm{d}p \, \mathrm{d}\zeta. \tag{4.5b}$$

Then, on substituting the leading-order solution (3.11) into the displacement (4.4), and fluctuating velocity fields (4.3), the non-zero components of (1.1) necessary to evaluate the drift for $s \neq 0$ are

$$\overline{\xi_j \breve{u}_{1,j}} = \epsilon^2 \frac{\omega^2}{\alpha} \left[-\frac{\cosh^2 \alpha(z+1)}{\sinh^2 \alpha} \overline{H \sin\beta} + \frac{\sinh^2 \alpha(z+1)}{\sinh^2 \alpha} \overline{I \cos\beta} \right.$$
$$\left. - \epsilon^s U' \frac{\sinh \alpha(z+1) \cosh \alpha(z+1)}{\sinh^2 \alpha} \overline{K \sin\beta} \right] + O(\epsilon^{2s+2}) \quad (s \neq 0), \tag{4.6a}$$

and

$$\overline{\xi_3\xi_3\overline{u}_{1,33}} = \epsilon^2\frac{\omega^2}{\alpha^2}\left[\epsilon^s\frac{\sinh^2\alpha(z+1)}{\sinh^2\alpha}U''\overline{I^2}\right] + O(\epsilon^{2s+2}) \quad (s \neq 0). \tag{4.6b}$$

Note that the restriction on s arises because (3.11) presumes $s \neq 0$.

Now, in accord with GLM theory, any appropriate average may be employed to evaluate the terms, $\overline{I\cos\beta}$, $\overline{H\sin\beta}$, $\overline{K\sin\beta}$ and $\overline{I^2}$. Here we take a temporal average over one wave period. Phillips (2001*b*) discusses the evaluation of integrals of the class (4.5) and their averaging in some detail and points out that finite t_0 necessitates secular behaviour. This is eliminated by requiring $t_0 \to -\infty$, which, as mentioned above, ensures that the waves are infinitesimal in their embryonic stages (see also Craik 1982*b*; Phillips 1998). Then provided $t_1 - t_0 \gg \omega^{-1}$, which ensures that $\mathrm{d}A/\mathrm{d}t = O(\epsilon^{2\lambda})$, $\lambda \geqslant 1$, and on integrating by parts, we have, for example,

$$\overline{H\sin\beta} = \frac{\omega}{2\pi}\int_{t_1}^{t_1+(2\pi/\omega)}\int_{t_0}^{\zeta} A(p)\cos(\alpha x-\omega p)\sin(\alpha x-\omega\zeta)\,\mathrm{d}p\,\mathrm{d}\zeta = -\frac{1}{2\omega}, \tag{4.7a}$$

with

$$\overline{I\cos(\beta)} = \frac{1}{2\omega}, \quad \overline{K\sin(\beta)} = -\frac{1}{2\omega^2} \quad \text{and} \quad \overline{I^2} = \frac{1}{2\omega^2}. \tag{4.7b}$$

Substitution of the averaged Lagrangian integrals (4.7) into (4.6) then leads to the drift as

$$d_1 + d_0 = \frac{\epsilon^2\omega}{2\alpha}\operatorname{csch}^2\alpha\left[\cosh 2\alpha(z+1) + \frac{\epsilon^s U'}{\omega}\sinh\alpha(z+1)\cosh\alpha(z+1) + \frac{\epsilon^s U''}{2\alpha\omega}\sinh^2\alpha(z+1)\right] + O(\epsilon^{2s+2}) \quad (s\neq 0),$$

where, for generality, d_0 is a reference value (see below). We may also write the drift in terms of wave properties $(\mathfrak{a}, \sigma, \mathfrak{K})$ as $d_1 = \mathfrak{a}^2\sigma\mathfrak{K}D(z;\alpha)$, yielding

$$D + D_0 = \tfrac{1}{2}\operatorname{csch}^2\alpha\left\{\cosh 2\alpha(z+1) + \vartheta\left[\alpha U'\sinh 2\alpha(z+1) - \tfrac{1}{2}[U'' - U''\cosh 2\alpha(z+1)]\right]\right\} + O(\epsilon^{2s}\vartheta), \tag{4.8}$$

where

$$\vartheta = \frac{\epsilon^s}{2\alpha^{3/2}\sqrt{\mathfrak{g}\tanh\alpha}}. \tag{4.9}$$

Accordingly,

$$\frac{\mathrm{d}D}{\mathrm{d}z} = \frac{1}{2}\operatorname{csch}^2\alpha\left\{2\alpha\sinh 2\alpha(z+1) + \vartheta\left[2\alpha U''\sinh 2\alpha(z+1) - \frac{1}{2}[U''' - (U''' + 4\alpha^2 U')\cosh 2\alpha(z+1)]\right]\right\}. \tag{4.10}$$

Of course, (4.8) must, in the limit $\vartheta \to 0$, recover Longuet-Higgins' shallow-water expression for Stokes drift D_s (Longuet-Higgins 1953) and it does, viz.

$$D + D_0 \sim D_s + D_{0s} = \frac{1}{2}\left[\operatorname{csch}^2\alpha\cosh 2\alpha(z+1) - \frac{\cot\alpha}{\alpha}\right], \tag{4.11}$$

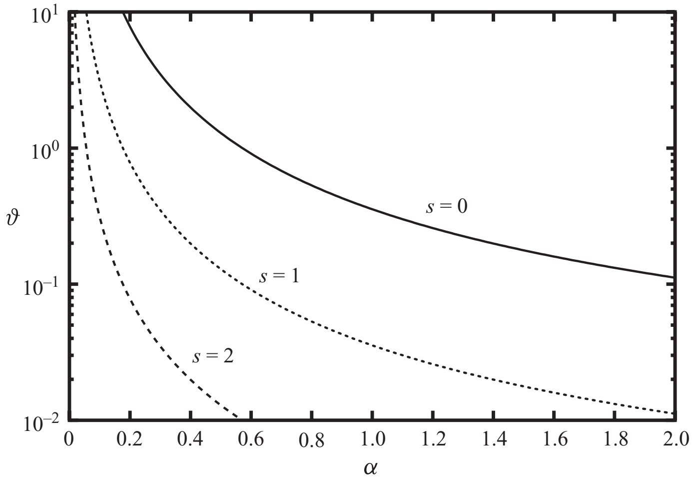

FIGURE 1. Plot of the parameter ϑ against α for $s = 0$, $s = 1$ and $s = 2$.

where the reference value D_{0s} is chosen here to ensure that the total horizontal mass transport in a channel is zero. We further see that deviations from D_s require that the parameter $\vartheta = O(1)$ (see (4.9)), which (since $\mathfrak{g}$ is the inverse square of the Froude number and $O(1)$) concurs with our finding in § 1 that shear plays a role whenever $\alpha \sim \epsilon^{s/2}$. Thus, the drift is modified by shear whenever: (i) $s = 0$ and the wavelength of the waves is of the order of the water depth, or (ii) $s = 1$ and the waves are long with respect to the water depth, that is, the waves are shallow-water waves. At a pinch, of course, we could include $s = 2$ in this category, but the waves would be very long relative to the water depth and we would likely violate the $\mathfrak{a}/\mathfrak{h} \ll 1$ criterion discussed in § 1. We plot (4.9), which holds for $s \in [0, 2]$, in figure 1, using $\mathfrak{g} = 2.6$ and $\epsilon = 0.1$ from Marmorino *et al.* (2005).

5. Base flow

To gain further insight into the distribution of drift, we require temporal distributions of velocity evident in coastal waters, namely profiles for wind- or stress-driven and current-driven shear flows. For completeness we derive both.

5.1. *Rayleigh stress problem in water of finite depth*

Rayleigh's problem, in which a rigid unbounded boundary is moved suddenly in its own plane in a viscous fluid of infinite extent, was extended by Batchelor (1967, § 4.3) to a fluid of finite extent. Here the problem is similar but the boundary condition on the moving boundary (the free surface $z = 0$) is different, namely constant stress (Neumann), as would be applied by wind at the free surface, rather than constant velocity (Dirichlet), while the other parallel rigid boundary (at $z = -1$) is held stationary.

Our problem is thus specified by (3.2) with $G = 0$, together with the following initial and boundary conditions, viz.

$$\frac{\partial U}{\partial z}(0, \tau) = 1, \quad U(-1, \tau) = 0 \quad \text{for} \quad \tau > 0, \tag{5.1a}$$

$$U(z, 0) = 0 \quad \text{for} \quad -1 \leqslant z < 0. \tag{5.1b}$$

Actually, the appropriate solution to (3.2) is found by first transforming to the new dependent variable

$$\mathscr{U}(z,\tau) = z + 1 - U(z,\tau), \tag{5.2}$$

which also satisfies (3.2) and from (5.1) has homogeneous boundary conditions at $z = 0$ and $z = -1$, viz.

$$\frac{\partial \mathscr{U}}{\partial z}(0,\tau) = 0, \quad \mathscr{U}(-1,\tau) = 0 \quad \text{for} \quad \tau > 0, \tag{5.3a}$$

with the initial condition

$$\mathscr{U}(z,0) = z + 1 \quad \text{for} \quad -1 \leqslant z < 0. \tag{5.3b}$$

Separation of variables then gives the solution

$$\mathscr{U}(z,\tau) = \sum_{n=0}^{\infty} A_n \exp\left[-\left(n+\frac{1}{2}\right)^2 \pi^2 \tau\right] \cos\left(n+\frac{1}{2}\right)\pi z,$$

which satisfies (5.3*a*); accordingly, the constants A_n are determined by invoking orthogonality of the eigenfunctions, which requires that

$$A_n = 2\int_{-1}^{0} (z+1)\cos\left(n+\frac{1}{2}\right)\pi z = \frac{8}{(2n+1)^2\pi^2}.$$

Then, by applying the transformation (5.2), the velocity profile for the stress Rayleigh problem in liquid of finite depth is given by

$$U(z,\tau) = (z+1) - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\exp\left[-\left(n+\frac{1}{2}\right)^2\pi^2\tau\right]\cos\left(n+\frac{1}{2}\right)\pi z. \tag{5.4}$$

Observe that $\partial U/\partial z \to 1$ as $\tau \to \infty$, indicating that the solution recovers Couette flow, as it must.

5.2. *Starting free-surface current-driven flow*

As our second candidate for the base flow we have a flow driven by a current for which the body force $G \neq 0$, with a rigid boundary at $z = -1$ and stress-free condition at the free surface $z = 0$. Then (3.2) is the governing equation with boundary and initial conditions given by

$$\frac{\partial U}{\partial z}(0,\tau) = 0, \quad U(-1,\tau) = 0 \quad \text{for} \quad \tau > 0,$$
$$U(z,0) = 0 \quad \text{for} \quad -1 \leqslant z \leqslant 0.$$

Here we first render (3.2) homogeneous by using as dependent variable the departure from its steady asymptotic limit. Then by methods which parallel that of the stress Rayleigh problem in § 5.1, the mean velocity is

$$U(z,\tau) = \frac{G}{2}\left[(z^2-1) + \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\exp\left[-\left(n+\frac{1}{2}\right)^2\pi^2\tau\right]\cos\left(n+\frac{1}{2}\right)\pi z\right]. \tag{5.5}$$

Profiles of the mean velocity given by (5.4) and (5.5) at various values of time τ are sketched in figures 2(*a*) and 2(*b*) respectively.

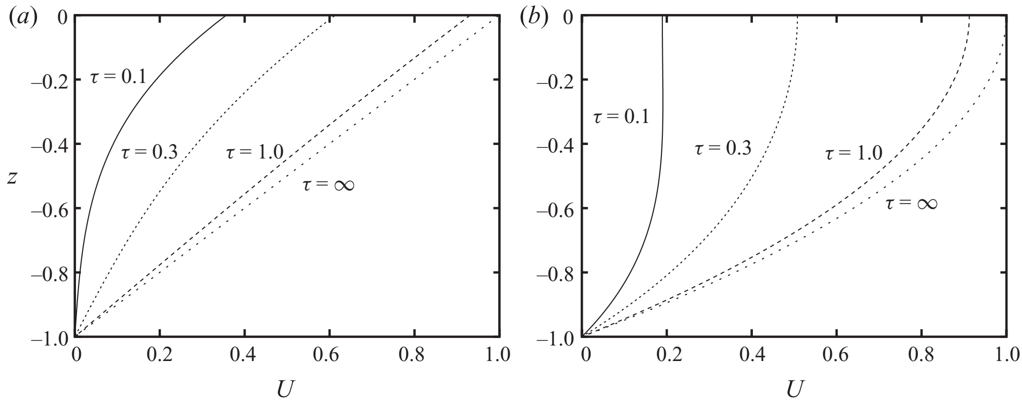

FIGURE 2. Temporal velocity profiles $U(z,\tau)$ for (*a*) shear-driven flow and (*b*) current-driven flow, at $\tau = 0.1, 0.3, 1.0, \infty$.

6. Discussion: the drift in a moderate shear flow

Returning now to the drift, it is evident from (4.8) that a good way to ascertain the role of shear on D is to compare D with the Stokes drift D_s (4.11). We begin by noting that D_s is affected by only one parameter α, while D is further affected by details of the shear flow, which, in turn, depends on τ. Note that because τ acts over a time scale significantly larger than the wave period we consider it a parameter. Further, because basic wave properties like α do not depend here upon current speed, we can consider it constant. So, to view our results, we consider various τ at fixed α in both stress-driven and current-driven shear. These are plotted in figure 3 with, for convenience, the offset D_{0s} (which should not be interpreted to mean the drift has a zero net transport). Of course, when $\alpha \gg 1$, we do not expect D to vary significantly from D_s (which is then exponential in z), because $\vartheta \ll 1$ (see figure 1), and in fact, this remains the case even when $\alpha = 1$. On the other hand, shear plays a significant role when $\alpha \ll 1$.

Looking first at figure 3(*a, c, e*), in which the shear flow is stress-driven, we see that while the variation with z is monotonic and generically not unlike that for D_s (which is parabolic in z when $\alpha \ll 1$), the difference between its maximum and minimum values, Λ, say, in cases *a*, *c* and *e*, exceeds that for D_s. This means that the level of mass transport at the free surface $z = 0$ and in the interior exceeds, and for *c* and *e* well exceeds, that for D_s, although to say precisely what the surface value $D(0)$ is, we need to calculate the offset D_0. Nevertheless, $D(0)$ remains in the direction of wave propagation, as is our mindset with surface drift.

Turning now to drift in current-driven shear (plots *b*, *d* and *f*), we find Λ values in cases *b* and *d* less than those for D_s, but larger in the case *f*. Moreover, as τ increases in the case *f*, the location of the maximum positive D moves from the free surface into the interior, which necessitates that the surface drift be oriented in a direction opposite to that of the wave propagation. This feature is not uncommon in drift fields in $s = 0$ shear (see Phillips & Wu 1994; Phillips & Shen 1996) and has interesting consequences with regard to the formation of Langmuir circulations in $s = 1$ shear. In this context of key interest is the gradient $\mathrm{d}D/\mathrm{d}z$ (4.10), which we plot in figure 4.

First, Langmuir circulations are excited by the CL type 2 instability mechanism (Craik & Leibovich 1976) in $s = 2$ shear only if the (spanwise-independent) differential drift is in the same sense as the shear. Phillips (2003) conjectures that this criterion

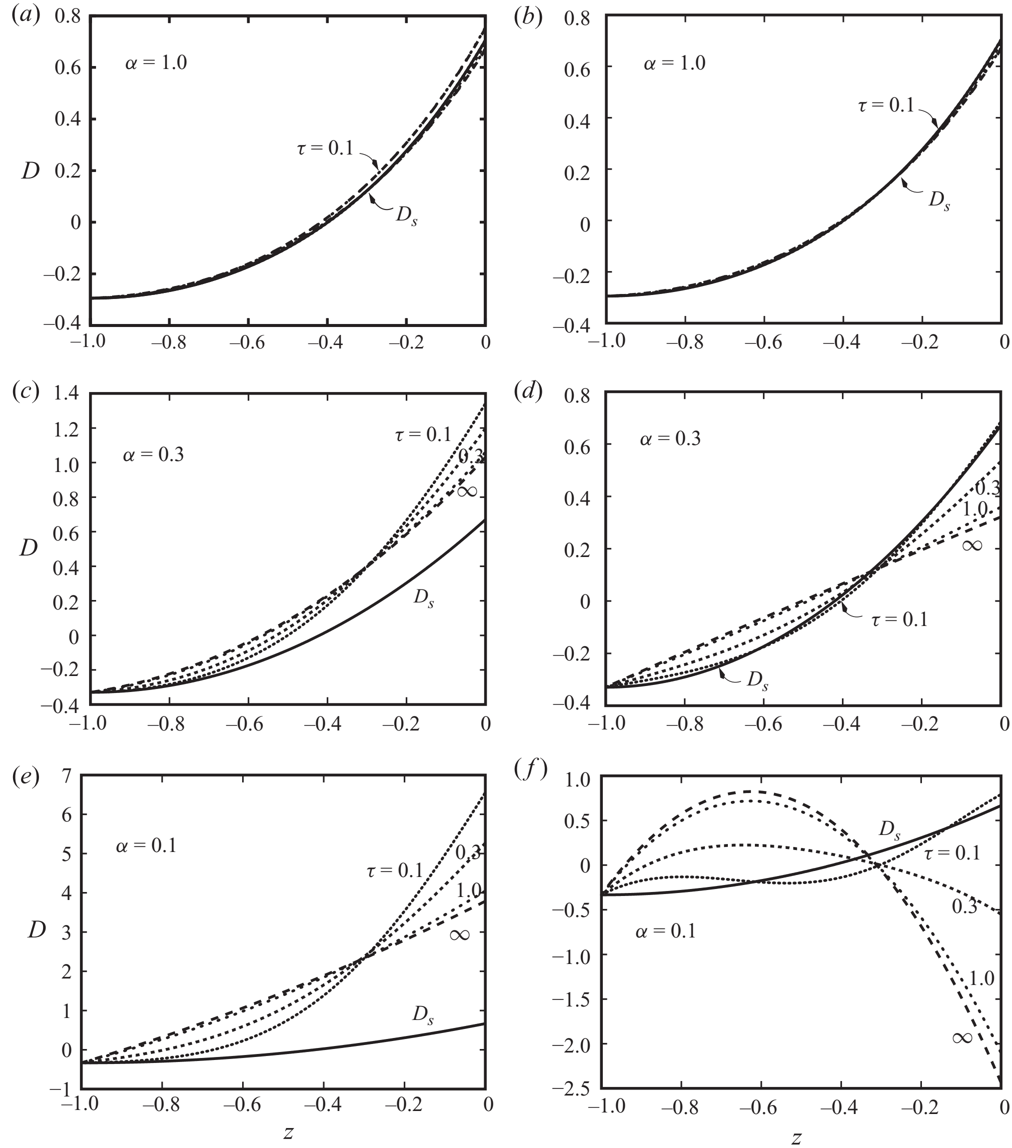

FIGURE 3. Plots of the drift with offset D_{s0} at $\alpha = 0.1$, 0.3, 1.0 for times $\tau = 0.1$, 0.3, 1.0, ∞, with the Stokes drift D_s for reference. (*a*, *c* and *e*) Stress-driven shear; (*b*, *d* and *f*) current-driven shear.

likely carries over to $s = 1$ shear, albeit subject to a difference in scaling. In strong ($s = 0$) shear, on the other hand, Langmuir circulations are excited by the CLg instability mechanism (Phillips 2003, 2005), provided the more intricate Craik–Phillips–Shen criterion (Craik 1982*c*; Phillips & Shen 1996; Phillips 2003) is satisfied.

Returning now to our findings, and assuming that Phillips' supposition for $s = 1$ shear is correct, we observe that although differential drift and shear are in the same sense (positive) throughout the domain in stress-driven shear (figure 4*a* and *c*), this is so only in the lower reaches of the layer for $\tau > 0.1$ in current-driven shear (plot *d*). Thus, while Langmuir circulations first form near the free surface in wind-driven

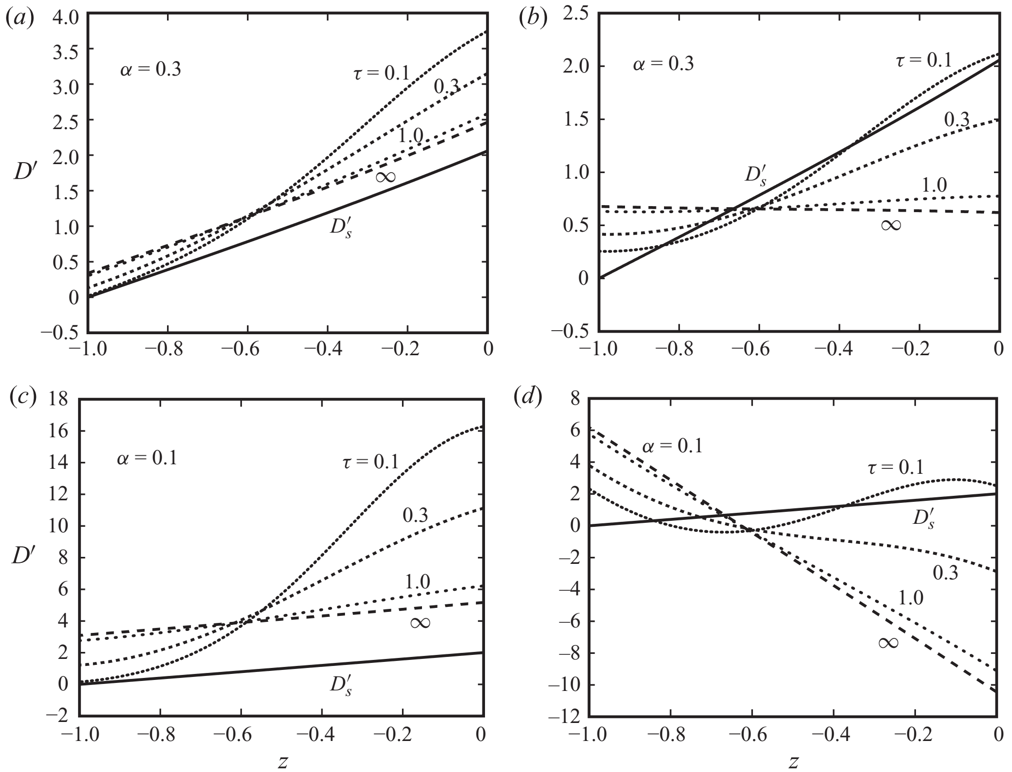

FIGURE 4. Plots of differential drift $D' = \mathrm{d}D/\mathrm{d}z$ at $\alpha = 0.1, 0.3$ for times $\tau = 0.1, 0.3, 1.0, \infty$, with differential Stokes drift D'_s for reference. (*a* and *c*) Stress-driven shear; (*b* and *d*) current-driven shear.

shear flows and grow downwards, top-down say, their counterparts in current-like flows form near the sea floor and grow upwards, that is bottom-up. In the latter case, however, the Langmuir circulations are confined to regions of the flow near the sea bed in which differential drift is positive. Thus, Langmuir supercells (Gargett *et al.* 2004), which, by definition, extend from the ocean surface to the sea floor, are likely a consequence of top-down, not bottom-up, Langmuir circulations. Accordingly, while bottom-up Langmuir circulations can play a role in scouring and redistributing particulates in coastal waters for their entire life cycle, their top-down counterparts can do the same only as Langmuir supercells.

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